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New results on the bijectivity of antipode of a Hopf algebra

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Abstract

Two results giving sufficient conditions for the bijectivity of the antipode of a Hopf algebra are proved.

0. Introduction

Let H be a Hopf algebra over a field k and $s: H \to H$ its antipode. A classical result due to Larson and Sweedler [4] says that s is bijective provided that $\dim H < \infty$. As was established by Radford [7], the same conclusion holds under the assumption that H^* contains nonzero integrals (that is, H is co-Frobenius). On the other hand, Takeuchi [13] constructed the free Hopf algebra $H(C)$ on a coalgebra C whose antipode is not bijective when C is the matrix coalgebra $\mathrm{Mat}_n(k)^*$, that is, the dual of the matrix algebra $\mathrm{Mat}_n(k)$, with $n > 1$. One might expect that the bijectivity of s depends on some finiteness conditions. The next result proved in the present paper uses purely ring-theoretic restrictions on H:

Theorem A.

(i) *If H is weakly finite then s is injective.*
(ii) *If H can be embedded into a left perfect ring Q such that Q is an essential extension of H as a right H-module, then s is bijective.*

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The class of weakly finite (also called stably finite) rings is very large (see, e.g., [9]). For instance, it includes all right noetherian rings. If a right noetherian ring R is semiprime then R embeds in the Goldie quotient ring Q as an essential submodule. More generally, assuming only that R is right nonsingular (meaning that no essential right ideal of R is the right annihilator of a nonzero element), we can take the Johnson–Utumi maximal quotient ring of R in place of Q. The ring Q is still semisimple artinian (see, e.g., [2, Theorem 3.17]), hence left perfect. We obtain

Corollary 1. *The antipode s of any right noetherian Hopf algebra H is injective. If H is, in addition, right nonsingular then s is bijective.*

There is another case where the existence of a quasi-Frobenius classical quotient ring was established by Wu and Zhang [15, Theorem 0.2(2)]. We are able therefore to answer a question contained in [15, Remark 3.9]:

Corollary 2. *Every noetherian affine PI Hopf algebra over a field has a bijective antipode.*

The above corollaries provide support for the following

Conjecture. *Noetherian Hopf algebras over a field have bijective antipode.*

A possible approach to this conjecture would be to verify that every right noetherian Hopf algebra H satisfies the hypotheses of Theorem A. We may regard H as a module algebra for the finite dual H° of H. If H is residually finite dimensional, then H contains no nontrivial H°-stable ideals. The existence of artinian classical quotient rings of right noetherian module algebras was investigated in [11]. Unfortunately we were unable to remove a restriction on Hopf algebras in that paper. As a result, the bijectivity of s is not clear even for residually finite-dimensional noetherian Hopf algebras.

Next we will look at the finite dual H° of an arbitrary Hopf algebra H. The Hopf algebra H° is always weakly finite, so that its antipode is injective by Theorem A.

Theorem B. *Denote by s° the antipode of H°.*

(i) *If k is either finite or an algebraic closure of a finite field then s° is bijective.*
(ii) *If k is a different field and H is the free Hopf algebra on the matrix coalgebra $C = \mathrm{Mat}_2(k)^*$ then s° is not surjective.*

The proof of part (ii) is based on ideas from [13]. However, we have to work with the algebra morphisms $H \to A$ only for finite-dimensional A. Over any field there exists a Hopf algebra whose antipode is not injective [6,14]. Schauenburg [10] gave also examples of Hopf algebras whose antipode is surjective but not injective.

1. Characterizations of injectivity and surjectivity

Throughout the paper we use standard definitions and notation from [5,12]. An indexed set of elements $c_{ij} \in H$, $1 \leqslant i, j \leqslant n$, will be called a *system of matrix counits* if

$$\Delta(c_{ij}) = \sum_{l=1}^{n} c_{il} \otimes c_{lj}, \qquad \varepsilon(c_{ij}) = \delta_{ij}$$

for all i, j. Such elements span a subcoalgebra of H. The latter is isomorphic to $\mathrm{Mat}_n(k)^*$ whenever c_{ij}'s are linearly independent.

With each system of matrix counits we can associate a matrix $X = (c_{ij}) \in \mathrm{Mat}_n(H)$. The relations

$$\sum_{l=1}^{n} c_{il} s(c_{lj}) = \delta_{ij} = \sum_{l=1}^{n} s(c_{il}) c_{lj}$$

show that X is invertible with inverse $X^{-1} = s(X)$ where we denote by $s(Y)$ the matrix obtained by applying s to all entries of $Y \in \mathrm{Mat}_n(H)$.

Let $\mathcal{M}^H$ and ${}^H\mathcal{M}$ be the categories, respectively, of right and left H-comodules. For each $U \in \mathcal{M}^H$ denote by $\rho : U \to U \otimes H$, $v \mapsto \sum_{(v)} v_{(0)} \otimes v_{(1)}$, the comodule structure map. If $e_1, \dots, e_n$ is a basis for U then the relations $\rho(e_j) = \sum_{i=1}^{n} e_i \otimes c_{ij}$ define a system of matrix counits. The linear span of those c_{ij}'s coincide with the smallest subcoalgebra $C \subset H$ satisfying $\rho(U) \subset U \otimes C$ [3, Proposition 1.4.3].

Conversely, every finite-dimensional subcoalgebra $C \subset H$ arises from a finite-dimensional comodule $U \in \mathcal{M}^H$ in this way. For instance, we may take $U = C$ with structure map $\rho = \Delta|_C$. In particular, C is spanned by a system of matrix counits.

Given $U \in \mathcal{M}^H$, denote by $U_s \in {}^H\mathcal{M}$ the comodule having the same underlying vector space as U and the structure map

$$U \to H \otimes U, \qquad v \mapsto \sum_{(v)} s(v_{(1)}) \otimes v_{(0)}.$$

When s is not bijective, the functor $\mathcal{M}^H \rightsquigarrow {}^H\mathcal{M}$ given by the assignment $U \mapsto U_s$ is not a category equivalence. This functor is nevertheless fully faithful by part (ii) of the next lemma.

Lemma 1.1.

(i) *Suppose that $U \in \mathcal{M}^H$ is a direct sum of vector subspaces V, W. Then V, W are subcomodules of U if and only if V, W are subcomodules of U_s.*
(ii) *If $V, W \in \mathcal{M}^H$ then the morphisms $V \to W$ in $\mathcal{M}^H$ coincide with the morphisms $V_s \to W_s$ in ${}^H\mathcal{M}$.*

Proof. (i)$_1$ Consider first a special case of (i) assuming $\dim U < \infty$. Pick a basis $e_1, \dots, e_n$ for U such that $e_1, \dots, e_m$ is a basis for V and $e_{m+1}, \dots, e_n$ a basis for W. Let $\rho(e_j) = \sum_{i=1}^{n} e_i \otimes c_{ij}$. Consider the block decomposition of the matrix

$$X = (c_{ij}) = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

corresponding to the partition of the index set as $\{1, \dots, m\} \cup \{m+1, \dots, n\}$. In order that V be a subcomodule of U (respectively U_s) it is necessary and sufficient that $c_{ij} = 0$ (respectively $s(c_{ij}) = 0$) for all $m < i \leqslant n$ and $1 \leqslant j \leqslant m$, which can be rewritten as $X_{21} = 0$ (respectively $s(X_{21}) = 0$). Similarly, W is a subcomodule of U (respectively U_s) if and only if $X_{12} = 0$ (respectively $s(X_{12}) = 0$). Suppose that $s(X_{21}) = 0$ and $s(X_{12}) = 0$ simultaneously. Then

$$s(X) = \begin{pmatrix} s(X_{11}) & 0 \\ 0 & s(X_{22}) \end{pmatrix}.$$

Since $s(X) = X^{-1}$, the matrix $s(X_{11})$ has to be the inverse of X_{11}, and then the equality $s(X_{11})X_{12} = 0$ yields $X_{12} = 0$. Similarly, $X_{21} = 0$. Consequently, V, W are subcomodules of U whenever they are subcomodules of U_s, while the opposite direction is obvious.

(ii) Since every comodule is a union of finite-dimensional subcomodules, the proof reduces to the case where $\dim V < \infty$, $\dim W < \infty$. Let $U = V \oplus W \in \mathcal{M}^H$ be the comodule direct sum, and identify V, W with subcomodules of U. For each linear map $f : V \to W$ consider its graph $\Gamma_f = \{(v, f(v)) \mid v \in V\}$ which is a vector subspace of U satisfying $U = \Gamma_f \oplus W$. Note that f is an $\mathcal{M}^H$-morphism if and only if Γ_f is a subcomodule of U. Similarly, for f to be an ${}^H\mathcal{M}$-morphism $V_s \to W_s$ it is necessary and sufficient that Γ_f be a subcomodule of U_s. The two conditions on Γ_f are equivalent by (i)$_1$.

(i)$_2$ Now we remove the restriction on $\dim U$ in (i). Let $p : U \to V$ be the projection with kernel W. Then p is an $\mathcal{M}^H$-morphism (respectively, an ${}^H\mathcal{M}$-morphism $U_s \to V_s$) if and only if V, W are subcomodules of U (respectively, U_s). The two conditions on p are equivalent by (ii). □

Lemma 1.2. *Let C be a subcoalgebra of H. The following properties are equivalent*:

(i) $s|_C : C \to H$ *is injective,*
(ii) $S \cap \operatorname{Ker} s = 0$ *for each simple subcoalgebra* $S \subset C$,
(iii) U_s *is a simple left H-comodule for each simple* $U \in \mathcal{M}^C$,
(iv) U *and* U_s *have the same subcomodules for each* $U \in \mathcal{M}^C$.

The category of right C-comodules $\mathcal{M}^C$ may be identified with a full subcategory of $\mathcal{M}^H$, so that $U_s \in {}^H\mathcal{M}$ makes sense for $U \in \mathcal{M}^C$.

Proof. (i) ⇒ (ii) is obvious.

(ii) ⇒ (iii). Given a simple $U \in \mathcal{M}^C$, there exists a simple subcoalgebra $S \subset C$ such that $\rho(U) \subset U \otimes S$. Suppose that V is a subcomodule of U_s. If $e_1, \ldots, e_n$ is a basis for U extending a basis $e_1, \ldots, e_m$ for V, then $\rho(e_j) = \sum_{i=1}^n e_i \otimes c_{ij}$ with $c_{ij} \in S$ for all $1 \leqslant i, j \leqslant n$ and we must have $s(c_{ij}) = 0$ for all $m < i \leqslant n$ and $1 \leqslant j \leqslant m$. Under the assumption $S \cap \operatorname{Ker} s = 0$ the equalities above imply that $c_{ij} = 0$ for i, j in the same range. This means that V is a subcomodule of U, whence either $V = 0$ or $V = U$.

(iii) ⇒ (iv). It suffices to consider the case $\dim U < \infty$ where we may proceed by induction on $\dim U$. Let W be a subcomodule of U_s. Suppose that $U \neq 0$ and V is any maximal subcomodule of U. Then U/V is simple in $\mathcal{M}^C$, so that $U_s/V_s \cong (U/V)_s$ is simple in ${}^H\mathcal{M}$ by (iii). In other words, V_s is a maximal subcomodule of U_s. By induction hypothesis V and V_s have the same subcomodules. In particular, $W \cap V$ has to be a subcomodule of U. If $W \subset V$ then $W = W \cap V$. Otherwise $W + V$ is a subcomodule of U_s properly containing V, whence $W + V = U$. In this case

$$U/(W \cap V) = V/(W \cap V) \oplus W/(W \cap V) \in \mathcal{M}^H,$$

where both summands are subcomodules of $(U/(W \cap V))_s$. As $W/(W \cap V)$ has to be a subcomodule of $U/(W \cap V)$ by Lemma 1.1, W is a subcomodule of U.

(iv) ⇒ (i). Since $\Delta(s(h)) = \sum_{(h)} s(h_{(2)}) \otimes s(h_{(1)})$ for all $h \in H$, the map s is an ${}^H\mathcal{M}$-morphism $H_s \to H$ where we regard H as either a right or a left comodule with respect to Δ. Hence

$\operatorname{Ker} s$ is a subcomodule of H_s, and so is $C \cap \operatorname{Ker} s$. Hypothesis (iv) applied with $U = C$ shows that $C \cap \operatorname{Ker} s$ is a right coideal of C. We deduce that $\sum_{(c)} s(c_{(1)}) \otimes c_{(2)} = 0$, and therefore

$$c = \sum_{(c)} \varepsilon(c_{(1)})c_{(2)} = \sum_{(c)} s(c_{(1)})c_{(2)}c_{(3)} = 0$$

for any $c \in C \cap \operatorname{Ker} s$. □

Lemma 1.3. *Let C be a subcoalgebra of H. The following properties are equivalent*:

(i) $C \subset s(H)$,
(ii) $s(H)$ *contains all simple subcoalgebras of* C.

Proof. (i) ⇒ (ii) is obvious.

(ii) ⇒ (i). We will prove by induction on $\dim U$ that $\rho(U) \subset U \otimes s(H)$ for every finite-dimensional $U \in \mathcal{M}^C$. Suppose that $U \neq 0$ and $V \subset U$ is any maximal subcomodule. Pick a basis $e_1, \ldots, e_n$ for U extending a basis $e_1, \ldots, e_m$ for V. Let $\rho(e_j) = \sum_{i=1}^n e_i \otimes c_{ij}$ with $c_{ij} \in C$. Use again the partition of indices $\{1, \ldots, m\} \cup \{m+1, \ldots, n\}$ to define the blocks of matrices

$$X = (c_{ij}) = \begin{pmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{pmatrix}, \qquad s(X) = \big(s(c_{ij})\big) = \begin{pmatrix} s(X_{11}) & s(X_{12}) \\ 0 & s(X_{22}) \end{pmatrix}.$$

Since $s(X) = X^{-1}$, we have $s(X_{22}) = X_{22}^{-1}$ and $X_{11}s(X_{12}) + X_{12}s(X_{22}) = 0$, whence

$$X_{12} = -X_{11}s(X_{12})X_{22}.$$

Since U/V is simple in $\mathcal{M}^C$, we have $\rho(U/V) \subset U/V \otimes S$ where $S \subset C$ is a simple subcoalgebra. By (ii) $S \subset s(H)$, so that X_{22} has entries in $s(H)$. By induction hypothesis $\rho(V) \subset V \otimes s(H)$, so that X_{11} has entries in $s(H)$. It follows that the same holds for X_{12} and the whole X. This proves the claim.

Every element $c \in C$ is contained in a finite-dimensional right coideal U of C. By the above $\Delta(c) \in U \otimes s(H)$. Hence $c = \sum_{(c)} \varepsilon(c_{(1)})c_{(2)} \in s(H)$. □

Note that the implications (ii) ⇒ (i) in Lemmas 1.2, 1.3 improve Radford's result [8, Proposition 3.5] which says that s is injective (respectively bijective) if and only if s is injective on the coradical H_0 of H (respectively s induces a bijective transformation of H_0).

2. Proof of Theorem A

We will need several properties of left perfect rings which are well known. The proofs are recalled below for the reader's convenience.

Lemma 2.1. *If R is a left perfect ring then*:

(i) *the ring* $\operatorname{Mat}_n(R)$ *is left perfect for any integer* $n > 0$,
(ii) *right regular elements of* $\operatorname{Mat}_n(R)$ *are invertible in* $\operatorname{Mat}_n(R)$,
(iii) R *is weakly finite, i.e., any equality* $XY = 1$ *in* $\operatorname{Mat}_n(R)$ *implies* $YX = 1$.

Proof. (i) A ring is defined to be left perfect if there exist projective covers in the category of left modules [1]. Clearly this property is Morita invariant.

(ii) As shown in [1] R satisfies the DCC on cyclic right ideals. Given $x \in R$ there exists therefore an integer $m > 0$ such that $x^m R = x^{m+1} R$. Hence $x^m = x^{m+1} y$ for some $y \in R$. If x is right regular, i.e., x has zero right annihilator, then we must have $xy = 1$. Furthermore, the equality $xyx = x$ implies that $yx = 1$ as well. In view of (i) we may replace R with $\mathrm{Mat}_n(R)$.

(iii) If $XY = 1$ then Y is right regular, whence $X = Y^{-1}$ by (ii). $\square$

Proof of Theorem A. (i) Let $U \in \mathcal{M}^H$ be a simple comodule. Suppose that V is a subcomodule of U_s. Pick any basis $e_1, \dots, e_n$ for U extending a basis $e_1, \dots, e_m$ for V. Write $\rho(e_j) = \sum_{i=1}^n e_i \otimes c_{ij}$ and

$$X = (c_{ij}) = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

where the blocks correspond to the partition of indices $\{1, \dots, m\} \cup \{m+1, \dots, n\}$. Clearly $s(X_{21}) = 0$. Since $s(X) = X^{-1}$, we have $X_{11} s(X_{11}) = 1$ in $\mathrm{Mat}_m(H)$ and $X_{21} s(X_{11}) = 0$. The first equality ensures that $s(X_{11})$ is invertible in $\mathrm{Mat}_m(H)$ by the weak finiteness of H. Then $X_{21} = 0$, so that V is a subcomodule of U. It follows that U_s is simple in ${}^H\mathcal{M}$, i.e., $C = H$ satisfies (iii) of Lemma 1.2.

(ii) By Lemma 2.1 Q is weakly finite, whence so too is H. Part (i) shows that s is injective. Let $C \subset H$ be any finite-dimensional subcoalgebra. It is spanned by a system of matrix counits c_{ij}, $1 \leqslant i, j \leqslant n$. Put $X = (c_{ij}) \in \mathrm{Mat}_n(H)$. We know already that $s(X) = X^{-1}$.

For each $Y \in \mathrm{Mat}_n(H)$ denote by Y^t the transpose of Y. As one checks straightforwardly, the assignment $Y \mapsto s(Y^t)$ defines an algebra antiendomorphism τ of $\mathrm{Mat}_n(H)$. It is clear that $\mathrm{Ker}\,\tau = 0$.

Let us check that X^t is right regular in $\mathrm{Mat}_n(H)$. Suppose that $X^t Y = 0$ for some $Y \in \mathrm{Mat}_n(H)$. Applying τ, we get $s(Y^t)s(X) = 0$, whence $s(Y^t) = 0$ because $s(X)$ is invertible in $\mathrm{Mat}_n(H)$. Now $Y = 0$ by the injectivity of τ.

For each $h \in H$ we denote by $D_h \in \mathrm{Mat}_n(H)$ the diagonal matrix all whose entries on the principal diagonal are equal to h and zero elsewhere. Then $ZD_h = (z_{ij}h)$ where $Z = (z_{ij}) \in \mathrm{Mat}_n(Q)$. If $Z \neq 0$ then there exists $h \in H$ such that $z_{ij}h \in H$ for all i, j and $z_{ij}h \neq 0$ for at least one pair of indices, that is, $0 \neq ZD_h \in \mathrm{Mat}_n(H)$. Indeed, $\mathrm{Mat}_n(Q)$ is an essential extension of $\mathrm{Mat}_n(H)$ as a right H-module. It has been proved that $X^t Z D_h \neq 0$, whence $X^t Z \neq 0$. In other words, X^t is right regular in $\mathrm{Mat}_n(Q)$. By Lemma 2.1 X^t is invertible in $\mathrm{Mat}_n(Q)$.

There exists $0 \neq h \in H$ such that the matrix $Y = (X^t)^{-1} D_h$ belongs to $\mathrm{Mat}_n(H)$. As $X^t Y = D_h$, an application of τ yields $D_{s(h)} = s(D_h^t) = s(Y^t)s(X)$, whence

$$D_{s(h)} X = s\bigl(Y^t\bigr) s(X) X = s\bigl(Y^t\bigr).$$

Thus all entries $s(h)c_{ij}$ of the matrix $D_{s(h)}X$ lie in $s(H)$, that is, $s(h)C \subset s(H)$.

Put $I = \{u \in H \mid s(u)C \subset s(H)\}$. We have just proved that $I \neq 0$. Let $u \in I$. If $h \in H$ then $s(uh)C = s(h)s(u)C \subset s(H)$. Hence $IH \subset I$. For any $c \in C$ we have

$$\sum s(u_{(2)})c_{(2)} \otimes s(u_{(1)})c_{(3)} \otimes c_{(1)} = \sum \Delta\bigl(s(u)c_{(2)}\bigr) \otimes c_{(1)} \in s(H) \otimes s(H) \otimes H,$$

whence

$$\sum s\bigl(s(u_{(1)})c\bigr) \otimes s^2(u_{(2)}) = \sum s\bigl(s(u_{(1)})c_{(3)}\bigr) \otimes c_{(1)} s\bigl(s(u_{(2)})c_{(2)}\bigr) \in s^2(H) \otimes H.$$

Suppose that $x \in H \otimes H$ is any element such that $(s \otimes s^2)(x) \in s^2(H) \otimes H$. Writing $x = \sum h_i \otimes e_i$ where the e_i's are linearly independent in H, we get $s(h_i) \in s^2(H)$ for each i since the elements $s^2(e_i)$ are linearly independent by the injectivity of s. Then also $h_i \in s(H)$, that is, $x \in s(H) \otimes H$. In particular,

$$\sum s(u_{(1)})c \otimes u_{(2)} \in s(H) \otimes H,$$

whence $\Delta(u) \in I \otimes H$. Thus I is a right ideal and a right coideal of H. The structure theorem for Hopf modules [12, Chapter IV] yields $I = H$ (it is easier to prove that I is a left coideal, but this does not give the required conclusion). The inclusion $1 \in I$ means that $C \subset s(H)$. As H is a union of its finite-dimensional subcoalgebras, we get $H = s(H)$. □

3. Finite duals

Denote by $\mathcal{F}$ the set of all ideals of finite codimension in H. The finite dual $H^\circ \subset H^*$ of H consists of all linear functions vanishing on an ideal in $\mathcal{F}$. The Hopf algebra H is said to be *residually finite dimensional* if the ideals in $\mathcal{F}$ have zero intersection.

Lemma 3.1.

(i) *If H is residually finite dimensional then s is injective.*
(ii) *The finite dual H° of any Hopf algebra is residually finite dimensional.*
(iii) *$H = s(H) + I$ for every $I \in \mathcal{F}$.*

Proof. (i) Every residually finite-dimensional algebra is weakly finite. In fact, if $XY = 1$ in $\mathrm{Mat}_n(H)$ then, denoting by $\pi_I : \mathrm{Mat}_n(H) \to \mathrm{Mat}_n(H/I)$ the canonical projection for $I \in \mathcal{F}$, we have $\pi_I(Y)\pi_I(X) = 1$ since $\pi_I(X)\pi_I(Y) = 1$ and the algebra $\mathrm{Mat}_n(H/I)$ is finite dimensional. Hence $YX - 1 \in \bigcap_{I\in\mathcal{F}} \operatorname{Ker} \pi_I = 0$. Thus Theorem A applies.

(ii) For each subcoalgebra $C \subset H$ the restriction map $H^\circ \hookrightarrow H^* \to C^*$ is an algebra morphism whose kernel $C^\perp$ is an ideal of H°. As $H^\circ/C^\perp$ is embedded in C^*, the ideal $C^\perp$ has finite codimension in H° whenever $\dim C < \infty$. The intersection of such ideals $C^\perp$ consists of all linear functions in H° which vanish on all finite-dimensional subcoalgebras of H. Since H coincides with the union of its finite-dimensional subcoalgebras, this intersection is zero.

(iii) Denote by s° the antipode of H°, so that $s^\circ(f) = f \circ s$ for $f \in H^\circ$. By (i) and (ii) s° is injective. Let $I \in \mathcal{F}$. If $s(H) + I \neq H$ then we can find $0 \neq f \in H^*$ vanishing on $s(H) + I$. Then $f \in H^\circ$ since $I \subset \operatorname{Ker} f$. On the other hand, $s^\circ(f) = 0$ since $s(H) \subset \operatorname{Ker} f$, a contradiction. □

Let A be an algebra and C a coalgebra. Following [13], we consider the convolution and twist convolution of linear maps $\varphi, \psi : C \to A$ defined by the rules

$$(\varphi * \psi)(c) = \sum_{(c)} \varphi(c_{(1)})\psi(c_{(2)}), \qquad (\varphi \times \psi)(c) = \sum_{(c)} \varphi(c_{(2)})\psi(c_{(1)}),$$

where $c \in C$. These are the multiplications in two convolution algebras $\mathrm{Hom}(C, A)$ and $\mathrm{Hom}(C^{\mathrm{cop}}, A)$ where C^{cop} is the opposite coalgebra. The unity element in both algebras is the map $c \mapsto \varepsilon(c)1$. We may now speak about $*$-invertible and $\times$-invertible linear maps $C \to A$.

Lemma 3.2.

(i) *Suppose $\varphi, \psi : H \to A$ are either $*$-inverses or $\times$-inverses of each other. Then φ is an algebra morphism if and only if ψ is an algebra antimorphism.*
(ii) *If $\varphi : H \to A$ is an algebra morphism then $\varphi \circ s$ is the $*$-inverse of φ.*
(iii) *If $\psi : H \to A$ is an algebra antimorphism then $\psi \circ s$ is the $\times$-inverse of ψ.*

Proof. Part (i) follows from [13, Lemma 25] or [3, Exercise 1.6.1]. Part (ii) is contained in [12, Lemma 4.0.3], and (iii) is similar to (ii). □

Lemma 3.3. *The following properties are equivalent*:

(i) *the antipode s° of H° is bijective,*
(ii) *for each $I \in \mathcal{F}$ there exists $J \in \mathcal{F}$ such that $I = s^{-1}(J)$,*
(iii) *for each maximal ideal $I \in \mathcal{F}$ there exists $K \in \mathcal{F}$ such that $s(I) + K \neq H$,*
(iv) *for each finite-dimensional algebra A every algebra morphism $\varphi : H \to A$ is $\times$-invertible.*

Proof. (i) $\Rightarrow$ (ii). Let $I \in \mathcal{F}$. As $\dim H/I < \infty$, we can find finitely many $f_1, \ldots, f_n \in H^*$ such that $I = \bigcap \mathrm{Ker}\, f_i$. Note that $f_i \in H^\circ$ for each i. Using the surjectivity of s° pick $g_1, \ldots, g_n \in H^\circ$ such that $f_i = g_i \circ s$ for each i. There exists $K \in \mathcal{F}$ contained in $\bigcap \mathrm{Ker}\, g_i$. Each f_i vanishes on $s^{-1}(K)$, so that $s^{-1}(K) \subset I$. As $H = s(H) + K$ by Lemma 3.1, s induces a bijection $H/s^{-1}(K) \to H/K$. The latter has to be an algebra antiisomorphism since s is an algebra antimorphism. Therefore there exists an ideal J of H such that $K \subset J$ and $s^{-1}(J) = I$. Clearly $J \in \mathcal{F}$.

(ii) $\Rightarrow$ (iii). If $I = s^{-1}(J)$ then $s(I) + J = J$. If I is, in addition, a maximal ideal then $1 \notin I$, whence $1 \notin J$, so that $J \neq H$. Take $K = J$.

(iii) $\Rightarrow$ (i). Each simple subcoalgebra of H° is of the form $I^\perp = \{f \in H^* \mid f(I) = 0\}$ for some maximal ideal $I \in \mathcal{F}$. Let K be as in (iii). Then $I + s^{-1}(K)$ is a proper ideal of H. Hence $I + s^{-1}(K) = I$, that is, $s^{-1}(K) \subset I$ by the maximality of I. As in the implication (i) $\Rightarrow$ (ii) we get $I = s^{-1}(J)$ for some $J \in \mathcal{F}$. Suppose that $f \in I^\perp$. Since s induces a linear injection $H/I \to H/J$, there exists $g \in H^*$ such that $g(J) = 0$ and $g \circ s = f$. Clearly $g \in H^\circ$. This proves that $I^\perp \subset s^\circ(H^\circ)$. We see that $s^\circ(H^\circ)$ contains all simple subcoalgebras of H°. By Lemma 1.3 s° is surjective and by Lemma 3.1 s° is injective.

(ii) $\Rightarrow$ (iv). Put $I = \mathrm{Ker}\,\varphi$. Under the hypotheses of (iv) $I \in \mathcal{F}$. By (ii) there exists $J \in \mathcal{F}$ such that $s^{-1}(J) = I$. Since $s(H) + J = H$ by Lemma 3.1, s induces an algebra antiisomorphism $H/I \to H/J$. Hence we can define an algebra antimorphism $\psi : H \to A$ such that $\mathrm{Ker}\,\psi = J$ and $\varphi = \psi \circ s$. By Lemma 3.2 ψ is the $\times$-inverse of φ.

(iv) $\Rightarrow$ (ii). Given $I \in \mathcal{F}$, let $\varphi : H \to H/I$ be the canonical projection. By (iv) φ has a $\times$-inverse $\psi : H \to H/I$. By Lemma 3.2 ψ is an algebra antimorphism and $\psi \circ s = \varphi$. Now $I = \mathrm{Ker}\,\varphi = s^{-1}(J)$ where we put $J = \mathrm{Ker}\,\psi \in \mathcal{F}$. □

Lemma 3.4. *Suppose that k_0 is a subfield of k and B_0 a k_0-subalgebra of a k-algebra B such that the k-linear map $B_0 \otimes_{k_0} k \to B$ extending the inclusion $B_0 \to B$ is bijective. If $x \in B_0$ is invertible in B then $x^{-1} \in B_0$.*

This is well known and easy to prove.

Lemma 3.5. *Suppose that the field k is either finite or an algebraic closure of a finite field. If $I \in \mathcal{F}$ and $C \subset H$ is a finite-dimensional subcoalgebra, then there exists an even integer $n > 0$ such that $s^{ni}(c) - c \in I$ for all $c \in C$ and all integers $i > 0$. If $|k| < \infty$ then one may take $n \leqslant 2|k|^{(\dim H/I)(\dim C)}$.*

Proof. Put $A = H/I$. There exist a finite subfield $k_0 \subset k$, a k_0-algebra A_0 and a k_0-coalgebra C_0 such that $A \cong A_0 \otimes_{k_0} k$ and $C \cong C_0 \otimes_{k_0} k$. To see this pick any basis $a_1, \ldots, a_m$ for A containing 1 and any basis $c_1, \ldots, c_t$ for C such that $\varepsilon(c_i)$ lies in the prime subfield of k for each i. Let $k_0 \subset k$ be any finite subfield containing all coefficients in the linear expansions of elements $a_i a_j$ and $\Delta(c_l)$ over the chosen basis for A and the basis $c_1 \otimes c_1$, $c_1 \otimes c_2, \ldots$ for $C \otimes C$. Take A_0 to be the k_0-linear span of $a_1, \ldots, a_m$ and C_0 the k_0-linear span of $c_1, \ldots, c_t$.

If $\varphi : C \to A$ is any linear map then we may also assume that $\varphi(C_0) \subset A_0$, enlarging k_0 if necessary. In fact k_0 has to contain all coefficients in the linear expansions of $\varphi(c_1), \ldots, \varphi(c_t)$. Let further φ be the restriction to C of the canonical projection $\pi : H \to A$, and choose k_0, A_0, C_0 as above. There are algebra isomorphisms

$$\mathrm{Hom}(C, A) \cong \mathrm{Hom}_{k_0}(C_0, A_0) \otimes_{k_0} k, \qquad \mathrm{Hom}\big(C^{\mathrm{cop}}, A\big) \cong \mathrm{Hom}_{k_0}\big(C_0^{\mathrm{cop}}, A_0\big) \otimes_{k_0} k.$$

For each $n \geqslant 0$ denote by $\varphi_n : C \to A$ the restriction of $\pi \circ s^n : H \to A$. Note that $\pi \circ s^n$ is an algebra morphism when n is even and an algebra antimorphism when n is odd. By Lemma 3.2 φ_{n+1} is the inverse of φ_n in one of the two convolution algebras above depending on whether n is even or odd. Lemma 3.4 and induction show that $\varphi_n(C_0) \subset A_0$ for all n. As $\mathrm{Hom}_{k_0}(C_0, A_0)$ is a vector space of dimension $(\dim A)(\dim C)$ over k_0, it contains $N = |k_0|^{(\dim A)(\dim C)}$ elements. It follows that the sequence of $N+1$ elements $\varphi_0, \varphi_2, \ldots, \varphi_{2N}$ contains at least two equal. Whenever $\varphi_i = \varphi_j$ for some $i, j > 0$ such that $j - i$ is even, we must have $\varphi_{i-1} = \varphi_{j-1}$ because φ_{i-1} is the inverse of φ_i and φ_{j-1} is the inverse of φ_j in one of the two convolution algebras considered above. Hence there exists an even integer $0 < n \leqslant 2N$ such that $\varphi_0 = \varphi_n$. Repeating the previous argument with the inverses, but now going upwards, we deduce $\varphi_i = \varphi_{n+i}$ for all $i \geqslant 0$. As a consequence, $\varphi_0 = \varphi_l$ whenever $l > 0$ is an integer multiple of n. But this means that $s^l(c) - c \in \operatorname{Ker} \pi = I$ for all $c \in C$. $\square$

We will say that H is *cogenerated* by a subset $\mathcal{X} \subset \mathcal{F}$ if no nonzero coideal of H is contained in all ideals $I \in \mathcal{X}$. This can be rephrased by saying that the subalgebra $A_{\mathcal{X}}$ generated by all subcoalgebras $I^{\perp} \subset H^{\circ}$ with $I \in \mathcal{X}$ is dense in H^*. Indeed, the set $\{h \in H \mid f(h) = 0$ for all $f \in A_{\mathcal{X}}\}$ is the largest coideal of H contained in all $I \in \mathcal{X}$. In particular, H is cogenerated by $\mathcal{F}$ if and only if H° is dense in H^*, which is a necessary and sufficient condition for H to be residually finite dimensional.

Proposition 3.6. *Suppose that k is finite.*

(i) *If H is generated by subcoalgebras of bounded dimension then for each $I \in \mathcal{F}$ there exists an integer $n > 0$ such that $s^n(h) - h \in I$ for all $h \in H$.*

(ii) *If H is cogenerated by ideals of bounded codimension then for each finite-dimensional subcoalgebra $C \subset H$ there exists an integer $n > 0$ such that $s^n|_C$ is the identity map. In particular, s is bijective.*

Proof. (i) Suppose H is generated by its subcoalgebras of dimension $\leqslant d$. There exists an even integer $n > 0$ such that $s^n(h) - h \in I$ for every $h \in H$ lying in such a subcoalgebra. By Lemma 3.5 the required inclusions are fulfilled provided that all positive integers up to $2|k|^{(\dim H/I)d}$ divide n. As the set $\{h \in H \mid s^n(h) \equiv h \pmod I\}$ is a subalgebra of H, it has to coincide with H.

(ii) Suppose H is cogenerated by its ideals of codimension $\leqslant e$. By Lemma 3.5 there exists an even integer $n > 0$ such that $s^n(c) - c \in I$ for all $c \in C$ and all ideals I of codimension $\leqslant e$. If $h = s^n(c) - c$ with $c \in C$ then

$$\Delta(h) = \sum_{(c)} \bigl(s^n(c_{(1)}) - c_{(1)}\bigr) \otimes s^n(c_{(2)}) + c_{(1)} \otimes \bigl(s^n(c_{(2)}) - c_{(2)}\bigr).$$

Hence $V = \{s^n(c) - c \mid c \in C\}$ is a coideal of H. As $V \subset I$ for each $I \in \mathcal{F}$ such that $\dim H/I \leqslant e$, we get $V = 0$. □

Question. *Suppose that k is finite and H is residually finite dimensional. Is then the antipode s necessarily bijective*?

4. Proof of Theorem B

Proof. (i) We will check condition (iv) of Lemma 3.3. Let $\varphi : H \to A$ be an algebra morphism where A is a finite-dimensional algebra. It suffices to show that $\varphi|_C$ is invertible in $\mathrm{Hom}(C^{\mathrm{cop}}, A)$ for each finite-dimensional subcoalgebra $C \subset H$. Put $I = \mathrm{Ker}\,\varphi$, and let n be as in Lemma 3.5. Then $\varphi \circ s^n$ agrees with φ on C. However, $\varphi \circ s^n$ has a $\times$-inverse $\psi = \varphi \circ s^{n-1}$ by Lemma 3.2 as $\psi : H \to A$ is an algebra antimorphism.

(ii) Denote by e_{11}, e_{12}, e_{21}, e_{22} the standard basis for C given by a linearly independent system of matrix counits. Let $\iota : C \to H$ denote the canonical embedding which identifies C with a subcoalgebra of H. Taking $A = \mathrm{Mat}_2(k)$, we will construct an algebra morphism $H \to A$ which violates condition (iv) of Lemma 3.3. Recall from [13, Proposition 4] that each sequence of linear maps $\varphi_n : C \to A$, $n = 0, 1, \ldots,$ such that φ_{2i+1} is the $*$-inverse of φ_{2i} and φ_{2i+2} is the $\times$-inverse of φ_{2i+1} for each $i \geqslant 0$ determines an algebra morphism $\varphi : H \to A$ such that $\varphi_n = \varphi \circ s^n \circ \iota$ for each n. For $\alpha, \beta, \gamma \in k$ define a linear map $\psi_{\alpha,\beta,\gamma} : C \to A$ by the assignments

$$e_{11} \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \qquad e_{12} \mapsto \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}, \qquad e_{21} \mapsto \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}, \qquad e_{22} \mapsto \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}.$$

Using the algebra isomorphisms

$$I : \mathrm{Hom}(C, A) \xrightarrow{\sim} \mathrm{Mat}_2(A), \qquad I(\xi) = \begin{pmatrix} \xi(e_{11}) & \xi(e_{12}) \\ \xi(e_{21}) & \xi(e_{22}) \end{pmatrix},$$

and

$$J : \mathrm{Hom}\bigl(C^{\mathrm{cop}}, A\bigr) \xrightarrow{\sim} \mathrm{Mat}_2(A), \qquad J(\xi) = \begin{pmatrix} \xi(e_{11}) & \xi(e_{21}) \\ \xi(e_{12}) & \xi(e_{22}) \end{pmatrix},$$

it is easy to figure out the inverses of $\psi_{\alpha,\beta,\gamma}$. Let us identify $\mathrm{Mat}_2(A)$ with $\mathrm{Mat}_4(k)$ and try to extend $\varphi_0 = \psi_{\alpha,\beta,\gamma}$ to a required sequence of maps. On the first step we have to invert the matrix

$$I(\varphi_0) = \begin{pmatrix} \alpha & 0 & 0 & \gamma \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ \gamma & 0 & 0 & \alpha \end{pmatrix}.$$

Thus the $*$-inverse φ_1 of φ_0 exists if and only if $(\alpha^2 - \gamma^2)\beta \neq 0$, in which case

$$\varphi_1 = \frac{1}{(\alpha^2 - \gamma^2)\beta} \psi_{\beta\alpha,\, \alpha^2-\gamma^2,\, -\beta\gamma}.$$

Next we have to invert the matrix

$$J(\varphi_1) = \frac{1}{(\alpha^2 - \gamma^2)\beta} \begin{pmatrix} \beta\alpha & 0 & 0 & 0 \\ 0 & \alpha^2-\gamma^2 & -\beta\gamma & 0 \\ 0 & -\beta\gamma & \alpha^2-\gamma^2 & 0 \\ 0 & 0 & 0 & \beta\alpha \end{pmatrix}.$$

We see that the $\times$-inverse φ_2 of φ_1 exists if and only if $((\alpha^2 - \gamma^2)^2 - \beta^2\gamma^2)\beta\alpha \neq 0$, in which case

$$\varphi_2 = \frac{\alpha^2 - \gamma^2}{((\alpha^2 - \gamma^2)^2 - \beta^2\gamma^2)\alpha} \psi_{\alpha',\beta',\gamma'},$$

where

$$\alpha' = \left(\alpha^2 - \gamma^2\right)^2 - \beta^2\gamma^2, \qquad \beta' = \left(\alpha^2 - \gamma^2\right)\beta\alpha, \qquad \gamma' = \beta^2\alpha\gamma.$$

Now the process repeats with different values of α, β, γ. Define three sequences of polynomials $f_n, g_n, h_n \in \mathbb{Z}[t]$ by recursive formulas

$$f_{n+1} = \left(f_n^2 - h_n^2\right)^2 - g_n^2 h_n^2, \qquad g_{n+1} = \left(f_n^2 - h_n^2\right) g_n f_n, \qquad h_{n+1} = g_n^2 f_n h_n$$

starting with $f_0 = t$, $g_0 = h_0 = 1$. By induction $\deg f_n = 4^n$, $\deg g_n, \deg h_n < 4^n$, and the leading coefficients of all polynomials are equal to 1. We can choose $\lambda \in k$ such that $f_n(\lambda) \neq 0$ and $g_n(\lambda) \neq 0$ for all n. When $\operatorname{char} k = 0$ any λ which is not an algebraic integer will do. When $\operatorname{char} k > 0$ take λ transcendental over the prime field. Specializing the initial values of α, β, γ to λ, 1, 1, we deduce that φ_0 extends to a desired sequence in which φ_{2i} is a scalar multiple of $\psi_{f_i(\lambda), g_i(\lambda), h_i(\lambda)}$ for each $i \geqslant 0$. This gives a certain algebra morphism $\varphi : H \to A$. Since the matrix

$$J(\varphi_0) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

is singular, φ_0 and φ are not $\times$-invertible. □

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